Machine learning over the free-parameters of the Black-Scholes equation: Stock market and Option market

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Abstract

The Black-Scholes equation is famous for predicting values for the prices of Options inside the stock market scenario. However, it has the limitation of depending on the estimated value for the volatility. On the other hand, several Machine learning techniques have been employed for predicting the values of the same quantity. In this paper we analyze some fundamental properties of the Black-Scholes equation and we then propose a way to train its free-parameters, the volatility in particular. This with the purpose of using this parameter as the fundamental one to be learned by a Machine Learning system and then improve the predictions in the stock market.

1. Introduction

The Black-Scholes equation was derived originally with the purpose of doing predictions about the fair price of an Option inside the stock market (1; 2; 3). The predictions were possible after designing a portfolio containing the option price and a derivative of it, such that the random fluctuations of the market, reflected inside the prices of the Options, get cancelled (2; 3). The cancellation of the random fluctuations then allows us to construct a differential equation, able to make the corresponding predictions. The Black-Scholes equation has two free-parameters, they are: 1). The interest rate. 2). The volatility. While the interest rate is normally given, the volatility is normally estimated and it is, in general, difficult to predict. Several studies about the BS equation and its properties have been done. In (2), The Hamiltonian formulation of the BS equation was done. In (4; 5), the symmetries of the BS Hamiltonian, as well as the mechanism of spontaneous symmetry breaking were analyzed. In (6), it was demonstrated that the Merton-Garman (MG) equation emerges naturally from the BS equation, after imposing the local symmetry conditions under changes of the prices of a stock. This is an important statement because the MG equation was created thinking on improving the predictions based on the fluctuations in volatility. The solutions of the BS equation, normally used for predicting the prices of Options, has an input variables the strike price of the Option, the time to maturity, price of the underlying security and the free-parameters. The interest rate and volatility also appear as parameters for the solution. Investors using the BS equation for doing predictions in the market, intend to estimate the volatility and then compare it with the Implied one (1). On the other hand, recent developments in Machine/Deep learning, allow us to make predictions by collecting data and then identifying patterns which can be used for learning parameters. In this paper we analyze some important details about the BS equation, and then we focus on the volatility parameter. If a machine learning system is created for doing predictions in the stock market, it should be able to tell the investors when to buy or sell and Option. For this purpose, it should be able to make predictions not only about the Implied value of the volatility, but also about the estimation based on different time scales.

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2. The Black Scholes equation

The Black-Scholes equation can be expressed in a Hamiltonian operator form as (2)

$$H_{BS}\psi(S,t)=E\psi(S,t). \tag{1}$$

Here H_{BS} is the Hamiltonian operator corresponding to the Black-Scholes equation. $\psi(S,t)$ is a function representing the prices of the Options as a function of the stock price S and of the time t. Explicitly, the BS Hamiltonian is given by

$$\hat{H}_{BS}=-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}+\left(\frac{1}{2}\sigma^2-r\right)\frac{\partial}{\partial x}+r, \tag{2}$$

where

$$S=e^x, \quad -\infty\leq x\leq\infty, \tag{3}$$

represents the prices of the stocks. In eq. (2), σ is the volatility and r is the interest rate. While the interest rate is easy to fix, the volatility is normally difficult to estimate (1). In fact, different persons could estimate different values for the volatility of the same Option under the same conditions (1). No matter what is the estimated value for the volatility, the dynamic of the BS system is determined by the relation between σ and r, which represent the free-parameters of the BS equation.

3. Symmetries and important parameters of the Black-Scholes equation

The symmetries of the BS equation have been analyzed previously in (4). Of particular interest, are those symmetries which are satisfied by the Hamiltonian but violated by the ground state. The ground state for the BS system, is defined as

$$\hat{H}_{BS}|S>=0, \tag{4}$$

which in the space of prices is just $<x|\hat{H}_{BS}|S>=0$, with x being the variable defined in eq. (3). It has been proved that while the Hamiltonian is invariant under changes of the prices of the stock, here represented as $\hat{p}=\frac{\partial}{\partial x}$, still this operator does not annihilate the ground state and then

$$\hat{p}|S>\neq 0. \tag{5}$$

This means that although $\hat{p}$ is a conserved operator, and then satisfying $[\hat{H}_{BS},\hat{p}]=0$ (Symmetry of the Hamiltonian), still $\hat{p}$ is not a symmetry of the ground state, which is reflected by the result (5). In physics, this type of phenomena is called Spontaneous Symmetry Breaking and it has several applications in condensed matter physics, material science and high energy physics (7). The consequences and interpretations of this phenomena inside the financial market, was carried out in (4; 5). Finally, an important result was derived recently, where it was demonstrated that the BS equation as well as the MG equation, are equivalent locally (6).

4. Equivalence between the Black-Scholes equation and the Merton-Garman: THe importance of volatility

We can now analyze the behavior of the BS Hamiltonian under local transformations involving changes of the prices of the stock. We can take a local transformation under the changes of prices as $U=e^{\omega\theta(x)}$. Here $\theta(x)$ is a variable depending on x, which also depends on the price of the stock S as the eq. (3) suggests. If the operator U were a symmetry of the system, then it would satisfy the condition $[\hat{H}_{BS},U]=0$ (8; 9). However, it is possible to demonstrate that this is not the case here after using the Hamiltonian given in eq. (2). In fact, if we use eq. (2), together with the definition of the local changes in price, then we have

$$[\hat{H}_{BS},U]\neq 0. \tag{6}$$

After some calculation, it is possible to demonstrate that in order to get an exact symmetry under local changes of the prices ($U=e^{\omega\theta(x)}$), then the BS Hamiltonian needs to add certain terms inside its definition in eq. (2). Under the action of the local transformation $U=e^{\omega\theta(x)}$, the BS Hamiltonian is changed as

$$\hat{H}_{BS}\to\hat{H}_{BS}+\frac{\sigma^2\omega(1+\omega)}{2}\left(\frac{\partial\theta(x)}{\partial x}\right)^2+ \\ \sigma^2\omega\left(\frac{\partial\theta(x)}{\partial x}\right)\frac{\partial}{\partial x}+\omega\left(\frac{1}{2}\sigma^2-r\right)\frac{\partial\theta(x)}{\partial x}. \tag{7}$$

Then the BS equation does not satisfy the gauge symmetry under changes of prices defined through the transformation $U=e^{\omega\theta(x)}$. For restoring the gauge-invariance, we have to extend the standard derivative in eq. (2), such that it becomes a covariant derivative. Without loss of generality, here we will define the covariant derivative as

$$\frac{\partial}{\partial x}\to\frac{\partial}{\partial x}+\hat{p}_y. \tag{8}$$

Here we interpret $\hat{p}_y$ as the momentum associated with the stochastic volatility. After replacing the ordinary derivative with the covariant derivative in eq. (2), we obtain

$$\hat{H}_{BS}\to\hat{H}=\frac{\sigma^2}{2}\left(-\hat{p}_x-\hat{p}_y\right)\left(\hat{p}_x+\hat{p}_y\right) \\ +\left(\frac{1}{2}\sigma^2-r\right)\left(\hat{p}_x+\hat{p}_y\right)+r. \tag{9}$$

The minus sign difference in the first term appears because the momentum associated to the changes of prices, as well as the momentum associated to the changes on the stochastic volatility are both non-Hermitian quantities, satisfying then the conditions

$$\hat{p}_x^+=\frac{\partial}{\partial x}^+=-\frac{\partial}{\partial x},\quad \hat{p}_y^+=\frac{\partial}{\partial y}^+=-\frac{\partial}{\partial y}. \tag{10}$$

Here the index $+$ means Hermitian conjugate operation. After an expansion, the equation (9) becomes

$$\hat{H}=-\frac{\sigma^2}{2}\hat{p}_x^2+\left(\frac{1}{2}\sigma^2-r\right)\hat{p}_x-\frac{\sigma^2}{2}\hat{p}_y^2-\sigma^2\hat{p}_x\hat{p}_y \\ +\left(\frac{1}{2}\sigma^2-r\right)\hat{p}_y+r. \tag{11}$$

The gauge invariance under a general transformation of the form $U=e^{\omega\theta(x,y)}$ for the new financial Hamiltonian defined in eq. (11) is guaranteed if the following conditions are satisfied

$$\left(\frac{\partial\theta}{\partial x}\right)^2=\frac{\omega}{1+\omega}\left(\frac{\partial\theta}{\partial y}\right)^2,$$
$$\left(\frac{\partial\theta}{\partial x}\right)\hat{p}_x=\left(\frac{\partial\theta}{\partial y}\right)\hat{p}_y,$$
$$\frac{\partial\theta}{\partial x}+\frac{\partial\theta}{\partial y}-4\frac{\partial^2\theta}{\partial x\partial y}=\frac{2r}{\sigma^2}\left(\frac{\partial\theta}{\partial x}+\frac{\partial\theta}{\partial y}\right). \quad (12)$$

These conditions are obtained after checking the invariance of eq. (11). In this way the changes due to local transformations of the new terms appearing in eq. (11), cancel exactly the additional terms appearing in eq. (7). Note that interestingly when $\sigma^2=2r$, then $\frac{\partial^2\theta}{\partial x\partial y}=0$. This condition corresponds, additionally, to the Hermiticity condition for the BS Hamiltonian. Independent of the values taken by the free-parameters of the system, as far as we can obtain the correct function $\theta(x,y)$, the gauge invariance of the Hamiltonian (11) is satisfied. The Hamiltonian defined in eq. (11) is the Merton-Garman Hamiltonian if we redefine the parameters appropriately as follows

$$\zeta^2=e^{-2y\left(\alpha-\frac{3}{2}\right)},$$
$$\rho\zeta=e^{-y\left(\alpha-\frac{3}{2}\right)},$$
$$r=\lambda e^{-y}+\mu. \quad (13)$$

These expressions guarantee the equivalence of the Hamiltonian in eq. (11) and the MG Hamiltonian (2). It is interesting to notice that the relations in eq. (13) give us the conditions $\rho=\pm1$, which are the extreme conditions for the parameter ρ. In the standard analysis of the MG equation, the parameter ρ respects the following condition

$$-1\leq\rho\leq1. \quad (14)$$

Then for the BS and the MG equations to be connected through gauge invariance under changes of the prices of the stock market system, as far as we define the covariant derivative as in eq. (8), then the MG parameter ρ can only take the extreme values. In this way, the white noises related to the time evolution of the stock price and volatility, satisfy the following conditions

$$<R_1(t)R_1(t')>=<R_2(t)R_2(t')>=\pm<R_1(t)R_2(t')>, \quad (15)$$

when the gauge invariance connects the BS and the MG equations (2; 4; 6). Finally, we must remark the interesting connection between the interest rate r and the volatility coefficients λ and μ inside eq. (13). In this way, when the BS and the MG equations are connected through the local symmetry transformations, the interest rate and the volatility are related through the parameters λ and μ.

4.1. The dynamical origin of the volatility

In order to analyze the dynamical origin of the volatility, we have to analyze the vacuum or ground state of the system. In (4), the general vacuum condition suggested a relation of the form

$$\phi_{yvac}=\left(\frac{\lambda e^{-y}+\mu-\frac{\zeta^2}{2}e^{2y(\alpha-1)}}{r-\frac{e^y}{2}}\right)\phi_{xvac}, \quad (16)$$

Under the assumptions done in this paper, this previous condition gives the result $\phi_{xvac}=\phi_{yvac}$. The result (16) is based on the general martingale state definition (2; 4). Although we could in principle work around the vacuum definition given in eq. (16), the appearance of the volatility inside the ground state definition, would make it difficult to visualize the mechanism behind the dynamical origin of the mass for the volatility. Then in this section, instead of considering the martingale state as a function of price (x) and volatility (y), we consider it as a function of the price only. Then we get

$$<x|\hat{V}(x,y)|S>=V(S)=-2\left(r-\frac{e^y}{2}\right)\phi_x\phi_y^2+r\phi_x^2\phi_y^2, \quad (17)$$

which ignores the term $<x|\hat{p}_y|S>=\partial S(x,t)/\partial y=0$ since in this special case, we are taking $S(x,t)$ (martingale state) as a state independent of y. Eq. (17) gives us the ordinary Martingale condition which is the same for the BS and MG cases. The ground state in eq. (17) is obtained from the condition $\partial V/\partial\phi_x=0$, obtaining then (4)

$$\phi_{vac}=1-\frac{\sigma^2}{2r}. \quad (18)$$

Then the field ϕ_x can be expanded around this ground state as

$$\phi(x)=\phi_{vac}+\bar{\phi}(x). \quad (19)$$

For understanding the effect of this field redefinition, we need to introduce the result (19) inside eq. (17). In this way, we get

$$<x|\hat{V}(x,y)|S>=V(S)=-2\left(r-\frac{e^y}{2}\right)(\phi_{vac}+\bar{\phi}(x))\phi_y^2+r(\phi_{vac}+\bar{\phi}(x))^2\phi_y^2. \quad (20)$$

From this expression, we obtain some terms of the form $\phi_{vac}\phi_y^2$ which represent the dynamical origin of the mass of the volatility field ϕ_y. More explicitly, the expression (20) becomes

$$< x|\hat{V}(x,y)|S >= V(S) = \left(-2\left(r-\frac{e^y}{2}\right)+r\phi_{vac}\right)\phi_{vac}\phi_y^2+.... \qquad (21)$$

Naturally, if ϕ_{vac} vanishes, then the massive term corresponding to the volatility field vanishes. This demonstrates that the dynamical origin of the volatility mass emerges from the relation between the parameters σ and r in eq. (18). Since in the MG equation $\sigma^2 = e^y$, then the phenomena is even more interesting than in standard situations because it involves non-linearities. Then the volatility field, generates its own mass dynamically because its influence appears inside the definition of the vacuum state ϕ_{vac}. Finally, since the terms in eq. (21) correspond to the second-order terms of the expansion of the price and volatility fields as it was done previously in (4; 5), then the kinetic terms in eq. (11), do not have contributions to the dynamical origin of the volatility mass, at least not at second-order. If we consider higher-order terms in the expansion, still the same arguments used for obtaining the result (21) work. The behavior of the kinetic terms for the MG equation, at all orders, was analyzed in (4).

Symbol	Name	Price (Intraday)	Change	% Change	Volume	Avg Vol (3 month)	Market Cap	PE Ratio (TTM)	52 Week Range
TSLA	Tesla, Inc.	249.73	-6.87	-2.68%	69.575M	137.145M	792.375B	74.40	101.81 314.67
LCID	Lucid Group, Inc.	5.98	+0.51	+9.32%	62.736M	25.824M	12.077B	N/A	5.46 21.78
CCL	Carnival Corporation & plc	14.02	-1.78	-11.23%	54.662M	38.113M	18.304B	N/A	6.11 16.40
AMD	Advanced Micro Devices, Inc.	109.04	-0.97	-0.88%	30.385M	67.448M	175.489B	573.55	54.57 132.83
PFE	Pfizer Inc.	36.57	-1.73	-4.51%	26.766M	24.786M	205.997B	7.20	36.16 54.93
MARA	Marathon Digital Holdings, Inc.	12.09	-0.62	-4.88%	26.333M	35.882M	2.045B	N/A	3.11 18.88
PLTR	Palantir Technologies Inc.	14.01	-0.02	-0.14%	23.925M	69.81M	29.805B	N/A	5.92 17.16
AMZN	Amazon.com, Inc.	129.86	+0.53	+0.41%	20.639M	62.762M	1.334T	295.43	81.43 146.57
NVDA	NVIDIA Corporation	414.51	-7.58	-1.80%	17.882M	46.844M	1.023T	217.97	108.13 439.90
NIO	NIO Inc.	8.51	+0.08	+0.93%	16.118M	51.474M	14.931B	N/A	7.00 24.43

Figure 1. Yahoo finance charts showing the values of certain variables for the most traded stocks. The Implied volatility has to match these values by following the BS equation (10).

Symbol	Underlying Symbol	Name	Strike	Expiration Date	Price (Intraday)	Change	% Change	Bid	Ask	Volume
IMPP230630P00002500	IMPP	IMPP Jun 2023 2.500 put	2.50	2023-06-29	0.1000	0.0000	0.00%	0.00	0.10	9
MNKD230630C00001000	MNKD	MNKD Jun 2023 1.000 call	1.00	2023-06-29	2.9500	0.0000	0.00%	2.50	5.50	2
RAD230630C00001000	RAD	RAD Jun 2023 1.000 call	1.00	2023-06-29	1.1000	0.0000	0.00%	0.55	3.10	1
UWMC230630C00001000	UWMC	UWMC Jun 2023 1.000 call	1.00	2023-06-29	4.5000	0.0000	0.00%	4.10	6.70	9
TSLL230630C00000500	TSLL	TSLL Jun 2023 0.500 call	0.50	2023-06-29	16.89	0.00	0.00%	15.70	16.00	1
EXPR230630P00001000	EXPR	EXPR Jun 2023 1.000 put	1.00	2023-06-29	0.3500	0.0000	0.00%	0.35	0.85	N/A
RIDE230630C00001500	RIDE	RIDE Jun 2023 1.500 call	1.50	2023-06-29	2.3000	0.0000	0.00%	1.00	3.50	N/A
UWMC230714C00001000	UWMC	UWMC Jul 2023 1.000 call	1.00	2023-07-13	4.6000	0.0000	0.00%	4.00	6.70	14
TLRY230630P00003000	TLRY	TLRY Jun 2023 3.000 put	3.00	2023-06-29	1.7200	0.0000	0.00%	1.37	1.57	4

Figure 2. Yahoo finance charts for Options with the highest Implied volatility (10).

5. Implied volatility vs. Estimated value of the volatility

After understanding the importance of the volatility inside the predictions of the prices of stocks and Options, in this section we proceed to explain how the volatility is estimated by the investors and then compared with its implied value. The implied value is normally the value of volatility matching with the data, like the one obtained from Yahoo-Finance (10). In other words, the Implied volatility is just the value of the volatility, such that after being substituted inside the BS equation, it brings out as a result the Option price observed in the market (11). For that purpose, we use the solutions of the BS equation, defined as

$$C = SN(d_1) - Ke^{-r(T-t)}N(d_2), \qquad (22)$$

for the Call Option. For the Put Option, analogue results are obtained (1). It can be proved that eq. (22) is a solution of the BS equation defined through the Hamiltonian (2) in eq. (1). In eq. (22), S is the stock price, K is the strike price, T is the maturity time for the European Call-Option and $N(d_1)$ and $N(d_2)$ are standard Normal distributions, which depend on the parameters d_1 and d_2. d_1 and d_2 are defined in (1) and they depend on the previously mentioned variables plus the volatility and interest rate. Note that eq. (22) cannot be solved trivially for the volatility σ and then iterative methods are applied for finding the Implied volatility. An example of the charts analyzed for getting the values of the Implied volatility can be seen in the figures (1) and (2). In general, once we know the value of the Implied volatility, we then compare with our estimated value of the volatility which could be based on daily, monthly or annual data. Once the Implied volatility, as well as the estimated volatility are expressed in the same time scale, then we compare their values. For understanding better this aspect, here we will illustrate an example taken from (1) and repeated here for clarifying the way how the investors decide how to decide whether or not they buy an Option. Suppose that the value of a European call Option on a non-dividend paying stock is 3,67 USD when S=33, $K = 30$, $r = 0,05$ and $T-t = 0,25$ years (three months). The Implied volatility would be the value of σ, such that when we introduce it inside the BS solution in eq. (22), together with the given values for S, K, r and $T-t$, gives us as a result the value for the Call Option $C = 3,67$ USD per share. It is not difficult to realize that the process is iterative. The result for this example, gives a volatility of 22% per annum. This result is taken from data of the charts, after solving σ in eq. (22). On the other hand, let's assume that the investor thinks that the volatility will be $1,5\%$ per day. Then we have to convert this daily estimation to an annual value. The formula to apply is

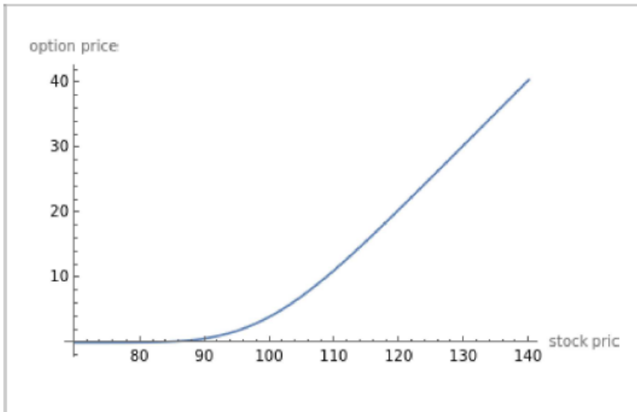

Figure 3. The relation between the prices of a Stock and the prices of an Option in agreement with the BS equation. Taken from (12).

$$\sigma_{annual} = \sigma_{day}\sqrt{D_{year}}, \tag{23}$$

where D_{year} is the number of trading days in a year. If we use this formula, then we get

$$\sigma_{annual} = 1,5\sqrt{256} = 24\%, \tag{24}$$

where we have assumed that there are 256 trading days in a year (Taking into account that the trading cannot be done everyday). From the standard theory we know that the higher the volatility of an Option is, the higher is its value. Since the investor estimates a volatility of $\sigma_{estimated} = 24\%$, which is a larger value than the estimated volatility, then he will certainly invest on the Option value. Then in general, if our estimated volatility is higher than the Implied volatility, then definitely we should buy the Option. On the other hand, if our estimation of volatility is smaller than the Implied value, then the investors should not buy the Option. The Machine Learning system will substitute the human part which estimate the values of volatility. The idea is to improve our capability for doing good decisions at the moment of investing. Then the difficulty of a Machine learning system, at the moment of predicting whether to invest or not, is on the comparison between the values taken by the different volatilities, namely, the estimated and the implied. In advance, there are some software and algorithms already working on these predictions, but still with some limitations. For instance, in the figure (3), it is illustrated the relation between the prices of an Option and the prices of the underlying stock. This relation depends on a series of variables and parameters, being the most difficult to deal with, the volatility.

6. Conclusions

In this summary paper we have introduced the model which we are using for analyzing the Implied volatility of the Option market for different stocks. The model is based on the BS equation and the decisions for investing or not over a stock, are based on comparisons over the estimated value of the volatility, which is in general different from the value taken by the implied volatility. The challenge of a machine learning system is to deal with this difference, being able to make predictions for both values of volatility, and then making subsequent comparisons. The main purpose of this paper, is to reduce the problem of supervised learning to a single parameter. In future research papers, we will explore the Machine learning techniques, ideal for improving the predictions in the market, based on the volatility values.

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